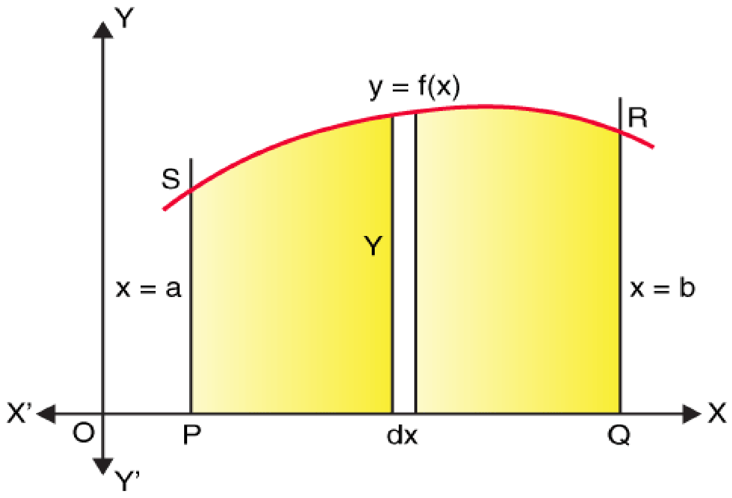
CLASS – 12

CHAPTER -8 Applications of Integrals

Application of Integration

One of the major application of integrals is in determining the area under the curves.

Consider a function y = f(x), then the area is given as



$A=\int_a^b dA=\int_a^b y.dx=\int_a^b f(x)dx$

Consider the two curves having equation of f(x) and g(x), the area between the region a,b of the two curves is given as-

dA = f(x) – g(x)]dx, and the total area A can be taken as-

$A=\int_a^b [f(x)–g(x)]dx$

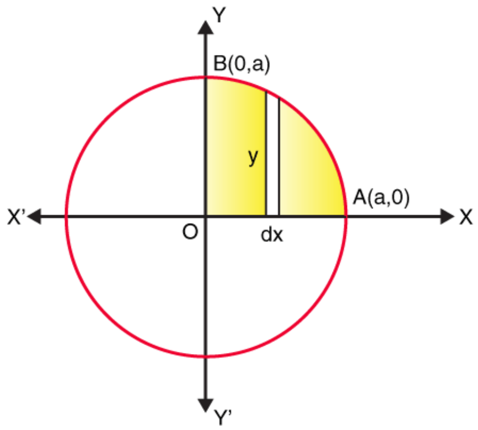
Application of Integrals Examples

Example 1:

Determine the area enclosed by the circle $x^2 + y^2 = a^2$

Solution:

Given, circle equation is $x^2 + y^2 = a^2$



From the given figure, we can say that the whole area enclosed by the given circle is as

= 4(Area of the region AOBA bounded by the curve, coordinates x=0 and x=a, and the x-axis)

As the circle is symmetric about both x-axis and y-axis, the equation can be written as

$= 4 \int_0^a y\, dx$ (By taking the vertical strips) ….(1)

From the given circle equation, y can be written as

$y = \pm\sqrt{(a^2-x^2)}$

As the region, AOBA lies in the first quadrant of the circle, we can take y as positive, so the value of y becomes $\sqrt{(a^2-x^2)}$

Now, substitute $y = \sqrt{(a^2-x^2)}$ in equation (1), we get

$= 4 \int_0^a \sqrt{(a^2-x^2)}\, dx$

Integrate the above function, we get

$= 4 \left[(x/2)\sqrt{(a^2-x^2)} + (a^2/2)\sin^{-1}(x/a)\right]_0^a$

Now, substitute the upper and lower limit, we get

$= 4[\{(a/2)(0)+(a^2/2)\sin^{-1} 1\}-\{0\}]$

$= 4(a^2/2)(\pi/2)$

$= \pi a^2$.

Hence, the area enclosed by the circle $x^2 + y^2 = a^2$ is πa^2.

Example 2:

Determine the area which lies above the x-axis and included between the circle and parabola, where the circle equation is given as $x^2+y^2 = 8x$, and parabola equation is $y^2 = 4x$.

Solution:

The circle equation $x^2+y^2 = 8x$ can be written as $(x-4)^2+y^2=16$. Hence, the centre of the circle is (4, 0), and the radius is 4 units. The intersection of the circle with the parabola $y^2 = 4x$ is as follows:

Now, substitute $y^2 = 4x$ in the given circle equation,

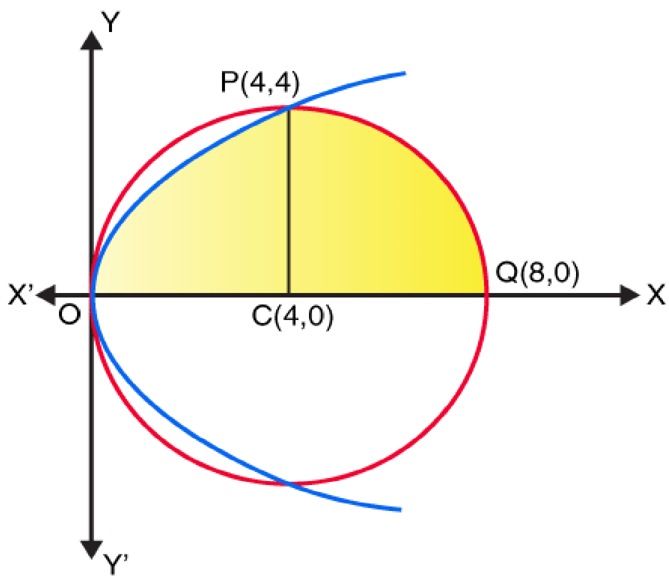
$x^2+4x = 8x$

$x^2 - 4x = 0$

On solving the above equation, we get

x=0 and x=4

Therefore, the point of intersection of the circle and the parabola above the x-axis is obtained as O(0,0) and P(4,4).



Hence, from the above figure, the area of the region OPQCO included between these two curves above the x-axis is written as

= Area of OCPO + Area of PCQP

$= \int_0^4 y dx + \int_4^8 y\, dx$

$= 2\ {}_0\int^4 \sqrt{x}\, dx + {}_4\int^8 \sqrt{[4^2 - (x-4)^2]}dx$

Now take x-4 = t, then the above equation is written in the form

$= 2\ {}_0\int^4 \sqrt{x}\, dx + {}_0\int^4 \sqrt{[4^2 - t^2]}dx$ …. (1)

Now, integrate the functions.

$2\ {}_0\int^4 \sqrt{x}\, dx = (2)(⅔)\ (x^3/2)_0{}^4$

$2\ {}_0\int^4 \sqrt{x}\, dx = 32/3$ …..(2)

${}_0\int^4 \sqrt{[4^2 - t^2]}dx = [(t/2)(\sqrt{[4^2-t^2]} + (½)(4^2)(\sin^{-1}(t/4)]_0{}^4$

${}_0\int^4 \sqrt{[4^2 - t^2]}dx = 4\pi$ …..(3)

Now, substitute (2) and (3) in (1), we get

$= (32/3) + 4\pi$

$= (4/3)\ (8+3\pi)$

Therefore, the area of the region that lies above the x-axis, and included between the circle and parabola is $(4/3)\ (8+3\pi)$.