CLASS – 12

CHAPTER -5 Continuity and Differentiability

Continuity

A function f(x) is said to be continuous at a point x = a, if
Left hand limit of f(x) at(x = a) = Right hand limit of f(x) at (x = a) = Value of f(x) at (x = a)
i.e. if at x = a, LHL = RHL = f(a)
where, LHL = $\lim_{x \to a^-} f(x)$ and RHL = $\lim_{x \to a^+} f(x)$
Note: To evaluate LHL of a function f(x) at (x = o), put x = a – h and to find RHL, put x = a + h.

Continuity in an Interval: A function y = f(x) is said to be continuous in an interval (a, b), where a < b if and only if f(x) is continuous at every point in that interval.

- Every identity function is continuous.
- Every constant function is continuous.
- Every polynomial function is continuous.
- Every rational function is continuous.
- All trigonometric functions are continuous in their domain.

Standard Results of Limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ (v) $\lim_{x \to \infty} \frac{1}{x^p} = 0, p \in (0, \infty)$ (vi) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(vii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ (viii) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$ (x) $\lim_{x \to 0} (1+x)^{1/x} = e$

(xi) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$ (xii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(xiii) $\lim_{x \to \infty} \sin x = \lim_{x \to \infty} \cos x =$ lies between – 1 to 1.

Algebra of Continuous Functions

Suppose f and g are two real functions, continuous at real number c. Then,

- f + g is continuous at x = c.
- f – g is continuous at x = c.
- f.g is continuous at x = c.
- cf is continuous, where c is any constant.
- (fg) is continuous at x = c, [provide g(c) ≠ 0]

Suppose f and g are two real valued functions such that (fog) is defined at c. If g is continuous at c and f is continuous at g (c), then (fog) is continuous at c.

If f is continuous, then |f| is also continuous.

Differentiability: A function f(x) is said to be differentiable at a point x = a, if Left hand derivative at (x = a) = Right hand derivative at (x = a)
i.e. LHD at (x = a) = RHD (at x = a), where Right hand derivative, where

Right hand derivative, $Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Left hand derivative, $Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$

Note: Every differentiable function is continuous but every continuous function is not differentiable.

Differentiation: The process of finding a derivative of a function is called differentiation.

Rules of Differentiation

Sum and Difference Rule: Let y = f(x) ± g(x).Then, by using sum and difference rule, it's derivative is written as

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$$

Product Rule: Let y = f(x) g(x). Then, by using product rule, it's derivative is written as

$$\frac{dy}{dx} = \left[\frac{d}{dx}(f(x))\right] g(x) + \left[\frac{d}{dx}(g(x))\right] f(x).$$

Quotient Rule: Let y = f(x)g(x); g(x) ≠ 0, then by using quotient rule, it's derivative is written as

$$\frac{dy}{dx} = \frac{g(x) \times \frac{d}{dx}[f(x)] - f(x) \times \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Chain Rule: Let y = f(u) and u = f(x), then by using chain rule, we may write

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \text{ when } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ both exist.}$$

Logarithmic Differentiation: Let y = $[f(x)]^{g(x)}$ ..(i)

So by taking log (to base e) we can write Eq. (i) as log y = g(x) log f(x). Then, by using chain rule

$$\frac{dy}{dx} = [f(x)]^{g(x)} \left[\frac{g(x)}{f(x)} f'(x) + g'(x) \log f(x) \right]$$

Differentiation of Functions in Parametric Form: A relation expressed between two variables x and y in the form x = f(t), y = g(t) is said to be parametric form with t as a parameter, when

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

(whenever dxdt≠0)

Note: dy/dx is expressed in terms of parameter only without directly involving the main variables x and y.

Second order Derivative: It is the derivative of the first order derivative.

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Some Standard Derivatives

(i) $\frac{d}{dx}(\sin x) = \cos x$

(ii) $\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(v) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(vi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(viii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(ix) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(x) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xi) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(xii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xiii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(xiv) $\frac{d}{dx}(\text{constant}) = 0$

(xv) $\frac{d}{dx}(e^x) = e^x$

(xvi) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

(xvii) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

Rolle's Theorem: Let f : [a, b] → R be continuous on [a, b] and differentiable on (a, b) such that f(a) = f(b), where a and b are some real numbers. Then, there exists at least one number c in (a, b) such that f'(c) = 0.

Mean Value Theorem: Let f : [a, b] → R be continuous function on [a, b]and differentiable on (a, b). Then, there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note: Mean value theorem is an expansion of Rolle's theorem.

Some Useful Substitutions for Finding Derivatives Expression

Expression	Substitution
(i) $a^2 + x^2$	$x = a\tan\theta$ or $x = a\cot\theta$
(ii) $a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
(iii) $x^2 - a^2$	$x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
(iv) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(v) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$