CLASS – 12

CHAPTER -4 Determinants

Determinant

Determinant is the numerical value of the square matrix. So, to every square matrix A = [a_{ij}] of order n, we can associate a number (real or complex) called determinant of the square matrix A. It is denoted by det A or |A|.

Note

(i) Read |A| as determinant A not absolute value of A.

(ii) Determinant gives numerical value but matrix do not give numerical value.

(iii) A determinant always has an equal number of rows and columns, i.e. only square matrix have determinants.

Value of a Determinant

Value of determinant of a matrix of order 2,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Value of determinant of a matrix of order 3,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A is given by expressing it in terms of second order determinant. This is known as expansion of a determinant along a row (or column).

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{(expansion along first row } R_1\text{)}$$

Note:-

(i) For easier calculations of determinant, we shall expand the determinant along that row or column which contains the maximum number of zeroes.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or -1 according to as (i + j) is even or odd.

Let A be a matrix of order n and let |A| = x. Then, $|kA| = k^n |A| = k^n x$, where n = 1, 2, 3,…

Minor: Minor of an element ay of a determinant, is a determinant obtained by deleting the i^{th} row and j^{th} column in which element ay lies. Minor of an element a_{ij} is denoted by M_{ij}.
Note: Minor of an element of a determinant of order n(n ≥ 2) is a determinant of order (n – 1).

Cofactor: Cofactor of an element a_{ij} of a determinant, denoted by A_{ij} or C_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of an element a_{ij}.
Note:-
(i) For expanding the determinant, we can use minors and cofactors as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

(ii) If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Singular and non-singular Matrix: If the value of determinant corresponding to a square matrix is zero, then the matrix is said to be a singular matrix, otherwise it is non-singular matrix, i.e. for a square matrix A, if |A| ≠ 0, then it is said to be a non-singular matrix and of |A| = 0, then it is said to be a singular matrix.

Theorems

(i) If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
(ii) The determinant of the product of matrices is equal to the product of their respective determinants, i.e. |AB| = |A||B|, where A and B are a square matrix of the same order.

Adjoint of a Matrix: The adjoint of a square matrix 'A' is the transpose of the matrix which obtained by cofactors of each element of a determinant corresponding to that given matrix. It is denoted by adj(A).
In general, adjoint of a matrix $A = [a_{ij}]_{n\times n}$ is a matrix $[A_{ji}]_{n\times n}$, where A_{ji} is a cofactor of element a_{ji}.

Properties of Adjoint of a Matrix

If A is a square matrix of order n × n, then

- $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj } (A^T) = (\text{adj } A)^T$

The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

NOTE: Since the area is a positive quantity we always take the absolute value of the determinant.

Properties of Determinants

To find the value of the determinant, we try to make the maximum possible zero in a row (or a column) by using properties given below and then expand the determinant corresponding that row (or column).

Following are the various properties of determinants:

1. If all the elements of any row or column of a determinant are zero, then the value of a determinant is zero.

2. If each element of any one row or one column of a determinant is a multiple of scalar k, then the value of the determinant is a multiple of k. then the value of the determinant is a multiple of k. i.e.

$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

3. If in a determinant any two rows or columns are interchanged, then the value of the determinant obtained is negative of the value of the given determinant. If we make n such changes of rows (columns) indeterminant Δ and obtain determinant Δ , then $\Delta_1 = (-1)^n \Delta$.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

4. If all corresponding elements of any two rows or columns of a determinant are identical or proportional, then the value of the determinant is zero.

$$\begin{vmatrix} a & b & c \\ b & e & f \\ a & b & c \end{vmatrix} = 0$$

[∴ R_1 and R_3 are identical.]

5. The value of a determinant remains unchanged on changing rows into columns and columns into rows. It follows that, if A is a square matrix, then |A'| = |A|.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

Note: det(A) = det(A'), where A' = transpose of A.

6. If some or all elements of a row or column of a determinant are expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants, i.e.

$$\begin{vmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix}.$$

7. In the elements of any row or column of a determinant, if we add or subtract the multiples of corresponding elements of any other row or column, then the value of determinant remains unchanged, i.e.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix} \ (C_1 \to C_1 + kC_2)$$

In other words, the value of determinants remains the same, if we apply the operation $R_i \to R_i + kE_j$ or $C_i \to C_j \to kC_j$.

Inverse of a Matrix and Applications of Determinants and Matrix

1. Inverse of a Square Matrix: If A is a non-singular matrix (i.e. |A| ≠ 0), then

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the inverse is $A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of A.

Note: Inverse of a matrix, if exists, is unique.

Properties of a Inverse Matrix

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1}=(A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- adj (A^{-1}) = (adj A$)^{-1}$

2. Solution of system of linear equations using inverse of a matrix.

Let the given system of equations be $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$.

We write the following system of linear equations in matrix form as AX = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Case I: If |A| ≠ 0, then the system is consistent and has a unique solution which is given by $X = A^{-1}B$.

Case II: If |A| = 0 and (adj A) B ≠ 0, then system is inconsistent and has no solution.

Case III: If |A| = 0 and (adj A) B = 0, then system may be either consistent or inconsistent according to as the system have either infinitely many solutions or no solutions.