CLASS – 12

CHAPTER -2 Inverse Trigonometric Functions

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Trigonometric functions are many-one functions but we know that inverse of function exists if the function is bijective. If we restrict the domain of trigonometric functions, then these functions become bijective and the inverse of trigonometric functions are defined within the restricted domain. The inverse of f is denoted by 'f^{-1}'.
Let y = f(x) = sin x, then its inverse is x = $\sin^{-1}$ y.

Domain and Range of Inverse Trigonometric Functions

Function	Domain	Range (Principal value branch)
$\sin^{-1} x$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	[-1, 1]	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\operatorname{cosec}^{-1} x$	$R-(-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
$\sec^{-1} x$	$R-(-1,1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$

$\sin^{-1}(\sin\theta) = \theta$; ∀ θ ∈ [−π2,π2]

$\cos^{-1}(\cos\theta) = \theta$; ∀ θ ∈ [0, π]

$\tan^{-1}(\tan\theta) = \theta$; ∀ θ [−π2,π2]

$\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = 0$; ∀ θ ∈ [−π2,π2] , θ ≠ 0

$\sec^{-1}(\sec\theta) = \theta$; ∀ θ ∈ [0, π], θ ≠ π2

$\cot^{-1}(\cot\theta) = \theta$; ∀ θ ∈ (0, π)

$\sin(\sin^{-1} x) = x$, ∀ x ∈ [-1, 1]

$\cos(\cos^{-1} x) = x; \forall x \in [-1, 1]$

$\tan(\tan^{-1}x) = x, \forall x \in R$

$\mathrm{cosec}(\mathrm{cosec}^{-1}x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

$\sec(\sec^{-1} x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$

$\cot(\cot^{-1} x) = x, \forall x \in R$

Note: $\sin^{-1}(\sin\theta) = \theta$; $\sin^{-1} x$ should not be confused with $(\sin x)^{-1} = 1/\sin x$ or $\sin^{-1} x = \sin^{-1}(1/x)$ land similarly for other trigonometric functions.

The value of an inverse trigonometric function, which lies in the range of principal value branch, is called the principal value of the inverse trigonometric function.
Note: Whenever no branch of an inverse trigonometric function is mentioned, it means we have to consider the principal value branch of that function.

Properties of Inverse Trigonometric Functions

(a) (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x; x \geq 1 \text{ or } x \leq -1$ (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x\ ;\ x \geq 1 \text{ or } x \leq -1$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x\ ;\ x > 0 \\ -\pi + \cot^{-1} x; x < 0 \end{cases}$

(b) (i) $\sin^{-1}(-x) = -\sin^{-1} x\ ; x \in [-1, 1]$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x\ ; x \in R$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x\ ;|x| \geq 1$

(c) (i) $\cos^{-1}(-x) = \pi - \cos^{-1} x\ ; x \in [-1, 1]$ (ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x\ ;|x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x\ ; x \in R$

(d) (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}; x \in [-1, 1]$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\ ; x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2};|x| \geq 1$

(e) (i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); xy > -1$

(f) (i) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right);|x| \leq 1$ (ii) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right); x \geq 0$

(iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1$

(iv) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}); \frac{-1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(v) $2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}); \frac{-1}{\sqrt{2}} \leq x \leq 1$ or $2\cos^{-1} x = \cos^{-1}(2x^2 - 1); 0 \leq x \leq 1$

(g) (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

(iii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2})$

(h) (i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3); \frac{-1}{2} \leq x \leq \frac{1}{2}$

(ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)\ ; \frac{1}{2} \leq x \leq 1$

(iii) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right); \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

(i) (i) $\sin^{-1} x = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(ii) $\cos^{-1} x = \sin^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

Following substitutions are used to write inverse trigonometric functions in simplest form:

S.No.	Expression	Substitute
1.	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
2.	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\ \text{cosec}\ \theta$
4.	$\sqrt{a + x}$ or $\sqrt{a - x}$	$x = a\cos\theta$ or $x = a\cos 2\theta$
5.	$\sqrt{1+x^2} \pm \sqrt{1-x^2}, \sqrt{\frac{1+x^2}{1-x^2}}, \sqrt{\frac{1-x^2}{1+x^2}}$	$x^2 = \cos^2\theta$
6.	$\sqrt{a^2+x^2} \pm \sqrt{a^2-x^2}, \sqrt{\frac{a^2+x^2}{a^2-x^2}}, \sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2\theta$
7.	$\sqrt{1+x} \pm \sqrt{1-x}, \sqrt{\frac{1-x}{1+x}}, \sqrt{\frac{1+x}{1-x}}$	$x = \cos 2\theta$
8.	$\sqrt{a+x} \pm \sqrt{a-x}, \sqrt{\frac{a+x}{a-x}}, \sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$

Remember Points

(i) Sometimes, it may happen, that some of the values of x that we find out does not satisfy the given equation.

(ii) While solving an equation, do not cancel the common factors from both sides.