CLASS – 12

CHAPTER -11 Three Dimensional Geometry

Direction Cosines of a Line: If the directed line OP makes angles α, β, and γ with positive X-axis, Y-axis and Z-axis respectively, then cos α, cos β, and cos γ, are called direction cosines of a line. They are denoted by l, m, and n. Therefore, l = cos α, m = cos β and n = cos γ. Also, sum of squares of direction cosines of a line is always 1,
i.e. $l^2 + m^2 + n^2 = 1$ or $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$
Note: Direction cosines of a directed line are unique.

Direction Ratios of a Line: Number proportional to the direction cosines of a line, are called direction ratios of a line.
(i) If a, b and c are direction ratios of a line, then l/a = m/b = n/c
(ii) If a, b and care direction ratios of a line, then its direction cosines are

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

(iii) Direction ratios of a line PQ passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ and direction cosines are

$$\frac{x_2 - x_1}{|\overrightarrow{PQ}|},$$

$$\frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|}.$$

Note:
(i) Direction ratios of two parallel lines are proportional.
(ii) Direction ratios of a line are not unique.

Straight line: A straight line is a curve, such that all the points on the line segment joining any two points of it lies on it.

Equation of a Line through a Given Point and parallel to a given vector $\vec{b}$
Vector form $\vec{r} = \vec{a} + \lambda \vec{b}$

where, $\vec{a}$ = Position vector of a point through which the line is passing
$\vec{b}$ = A vector parallel to a given line

Cartesian form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where, (x_1, y_1, z_1) is the point through which the line is passing through and a, b, c are the direction ratios of the line.
If l, m, and n are the direction cosines of the line, then the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

Remember point: Before we use the DR's of a line, first we have to ensure that coefficients of x, y and z are unity with a positive sign.

Equation of Line Passing through Two Given Points
Vector form: $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, $\lambda \in R$, where a and b are the position vectors of the points through which the line is passing.

Cartesian form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where, (x_1, y_1, z_1) and (x_2, y_2, z_2) are the points through which the line is passing.

Angle between Two Lines
Vector form: Angle between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given as

$$\cos\theta = \left| \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}| \cdot |\vec{b_2}|} \right|$$

Cartesian form If θ is the angle between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$, then $\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$

or $$\sin\theta = \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Also, angle (θ) between two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

or $$\sin\theta = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$$

Condition of Perpendicularity: Two lines are said to be perpendicular, when in vector form b1→·b2→=0; in cartesian form $a_1a_2 + b_1b_2 + c_1c_2 = 0$

or $l_1l_2 + m_1m_2 + n_1n_2 = 0$ [direction cosine form]

Condition that Two Lines are Parallel: Two lines are parallel, when in vector form b1→·b2→=|||b1→|||||b2→|||; in cartesian form a1/a2=b1/b2=c1/c2
or
l1/l2=m1/m2=n1/n2
[direction cosine form]

Shortest Distance between Two Lines: Two non-parallel and non-intersecting straight lines, are called skew lines.
For skew lines, the line of the shortest distance will be perpendicular to both the lines.
Vector form: If the lines are r⃗ =a1→+λb1→ and r⃗ =a2→+λb2→. Then, shortest distance

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

where a2→, a1→ are position vectors of point through which the line is passing and b1→, b2→ are the vectors in the direction of a line.

Cartesian form: If the lines are

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}.$$

Then, shortest distance,

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

Distance between two Parallel Lines: If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$, then the distance between parallel lines is

$$\left|\frac{\vec{b}\times(\vec{a_2}-\vec{a_1})}{|\vec{b}|}\right|$$

Note: If two lines are parallel, then they both have same DR's.

Distance between Two Points: The distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Mid-point of a Line: The mid-point of a line joining points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Plane: A plane is a surface such that a line segment joining any two points of it lies wholly on it. A straight line which is perpendicular to every line lying on a plane is called a normal to the plane.

Equations of a Plane in Normal form

Vector form: The equation of plane in normal form is given by $\vec{r}\cdot\vec{n}=d$, where $\vec{n}$ is a vector which is normal to the plane.

Cartesian form: The equation of the plane is given by ax + by + cz = d, where a, b and c are the direction ratios of plane and d is the distance of the plane from origin.

Another equation of the plane is lx + my + nz = p, where l, m, and n are direction cosines of the perpendicular from origin and p is a distance of a plane from origin.
Note: If d is the distance from the origin and l, m and n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

Equation of a Plane Perpendicular to a given Vector and Passing Through a given Point
Vector form: Let a plane passes through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{n}$, then $(\vec{r}-\vec{a})\cdot\vec{n}=0$
This is the vector equation of the plane.
Cartesian form: Equation of plane passing through point (x_1, y_1, z_1) is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where, a, b and c are the direction ratios of normal to the plane.

Equation of Plane Passing through Three Non-collinear Points
Vector form: If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of three given points, then equation of a plane passing through three non-collinear points is $(\vec{r}-\vec{a})\cdot\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0$.
Cartesian form: If (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3) are three non-collinear points, then equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

If above points are collinear, then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0.$$

Equation of Plane in Intercept Form: If a, b and c are x-intercept, y-intercept and z-intercept, respectively made by the plane on the coordinate axes, then equation of plane is $x/a+y/b+z/c=1$

Equation of Plane Passing through the Line of Intersection of two given Planes
Vector form: If equation of the planes are $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then equation of any plane passing through the intersection of planes is

$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$
where, λ is a constant and calculated from given condition.
Cartesian form: If the equation of planes are $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, then equation of any plane passing through the intersection of planes is $a_1x + b_1y + c_1z – d_1 + \lambda (a_2x + b_2y + c_2z – d_2) = 0$
where, λ is a constant and calculated from given condition.

Coplanarity of Two Lines
Vector form: If two lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, then
$(\vec{a_2}-\vec{a_1})\cdot(\vec{b_2}-\vec{b_1})=0$

Cartesian form If two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$

Angle between Two Planes: Let θ be the angle between two planes.
Vector form: If $\vec{n_1}$ and $\vec{n_2}$ are normals to the planes and θ be the angle between the planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then θ is the angle between the normals to the planes drawn from some common points.

$$\cos\theta=\left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\ |\vec{n_2}|}\right|.$$

Note: The planes are perpendicular to each other, if $\vec{n_1}\cdot\vec{n_2}=0$ and parallel, if $\vec{n_1}\cdot\vec{n_2}=||\vec{n_1}||\,||\vec{n_2}||$
Cartesian form: If the two planes are $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Note: Planes are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and planes are parallel, if $a_1a_2=b_1b_2=c_1c_2$

Distance of a Point from a Plane
Vector form: The distance of a point whose position vector is $\vec{a}$ from the plane
$\vec{r}\cdot\hat{n}=d$ is $|d-\vec{a}\,\hat{n}|$

Note:
(i) If the equation of the plane is in the form $\vec{r}\cdot\vec{n}=d$, where $\vec{n}$ is normal to the plane, then the perpendicular distance is $||\vec{a}\cdot\vec{n}-d|||\vec{n}||$
(ii) The length of the perpendicular from origin O to the plane $\vec{r}\cdot\vec{n}=dis|d|||\vec{n}||$ [$\because \vec{a}=0$]

Cartesian form: The distance of the point (x_1, y_1, z_1) from the plane Ax + By + Cz = D is

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$$

Angle between a Line and a Plane
Vector form: If the equation of line is $\vec{r}=\vec{a}+\lambda\vec{b}$ and the equation of plane is $\vec{r}\cdot\vec{n}=d$, then the angle θ between the line and the normal to the plane is

$$\cos\theta=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

and so the angle Φ between the line and the plane is given by 90° – θ,
i.e. sin(90° – θ) = cos θ

$$\sin\phi=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Cartesian form: If a, b and c are the DR's of line and lx + my + nz + d = 0 be the equation of plane, then

$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}.$$

If a line is parallel to the plane, then al + bm + cn = 0 and if line is perpendicular to the plane, then al=bm=cn

Remember Points
(i) If a line is parallel to the plane, then normal to the plane is perpendicular to the line. i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$
(ii) If a line is perpendicular to the plane, then DR's of line are proportional to the normal of the plane.
i.e. a1/a2=b1/b2=c1/c2
where, a_1, b_1 and c_1 are the DR's of a line and a_2, b_2 and c_2 are the DR's of normal to the plane.